Euler- Poincaré Characteristic,
in the frontier between topology, geometry and combinatorics

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Summary of the course

1 – **Euler-Poincaré characteristic and the classification of compact – surfaces.**

2/3 – **Consequences and applications of the classification theorem and Euler - Poincaré characteristic.**

Day 1:Euler-Poincaré characteristic and the classification of compact surfaces

- Compact surfaces as labelled 2n-polygons/words
- Sketch of a proof of the classification Theorem

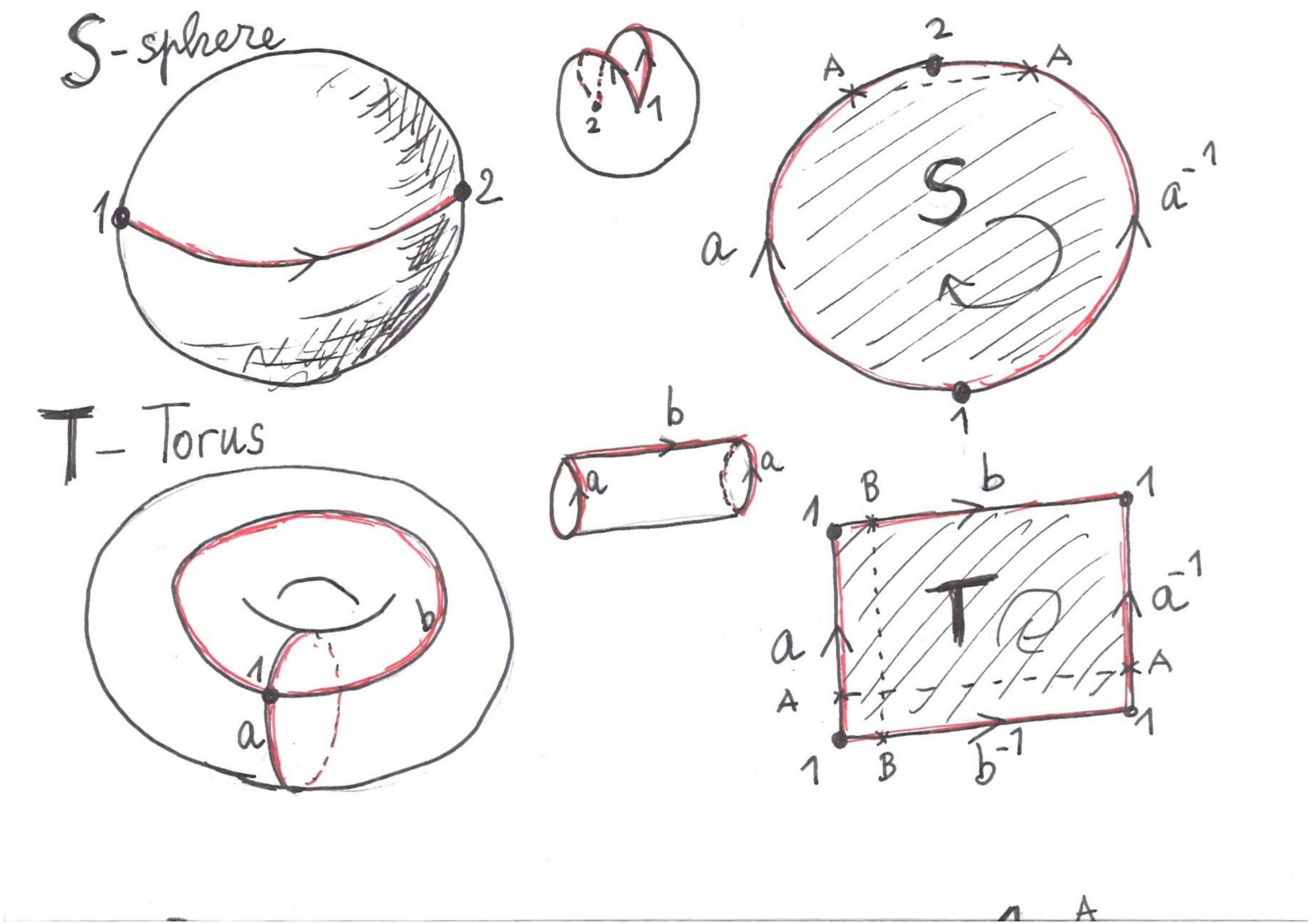
What is a (compact) surface

- **Surface = topological space + connected +compact +Hausdorff + locally plane**

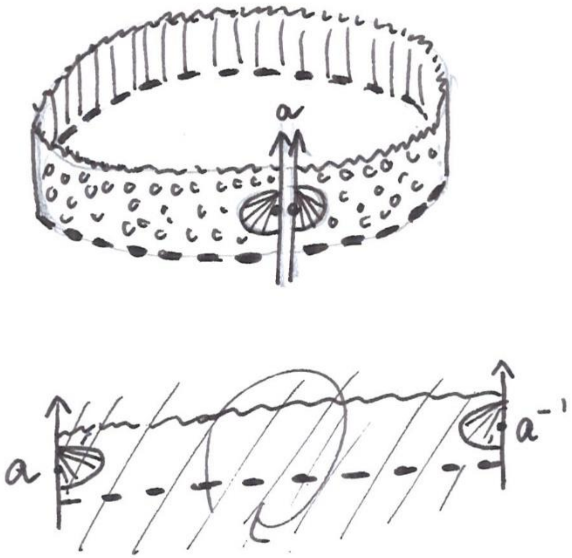
2- dimensional compact manifold with no boundary

- Connected - just one piece (not a union of disjoint open sets)
- Compact - every covering by open sets contains a finite covering
- Haudorff = every pair of points have non intersecting neighbourhoods
- **Locally plane** =every point has a neighbourhood homeomorphic to an open disc of the (real euclidean) plane.

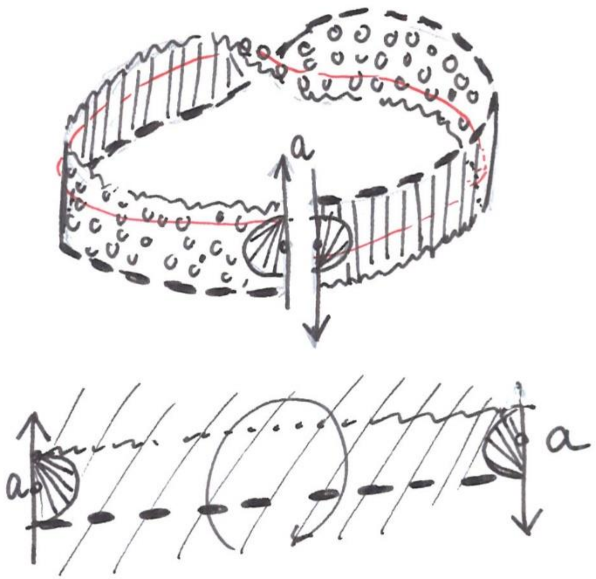
Plane models of surfaces : Examples



TWO WAYS OF CUTTING AND PASTING DISCS



BAND



MÖBIUS BAND

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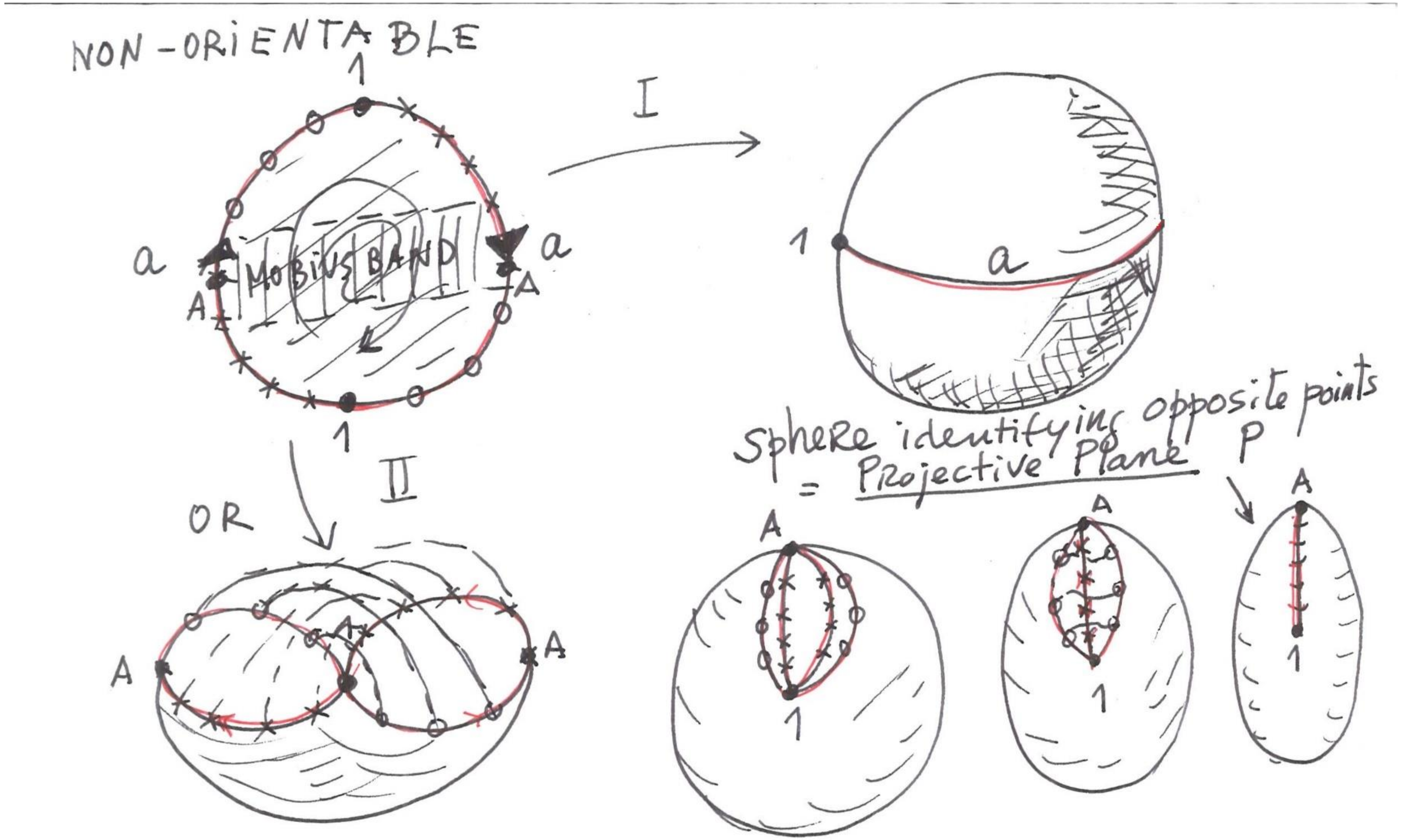
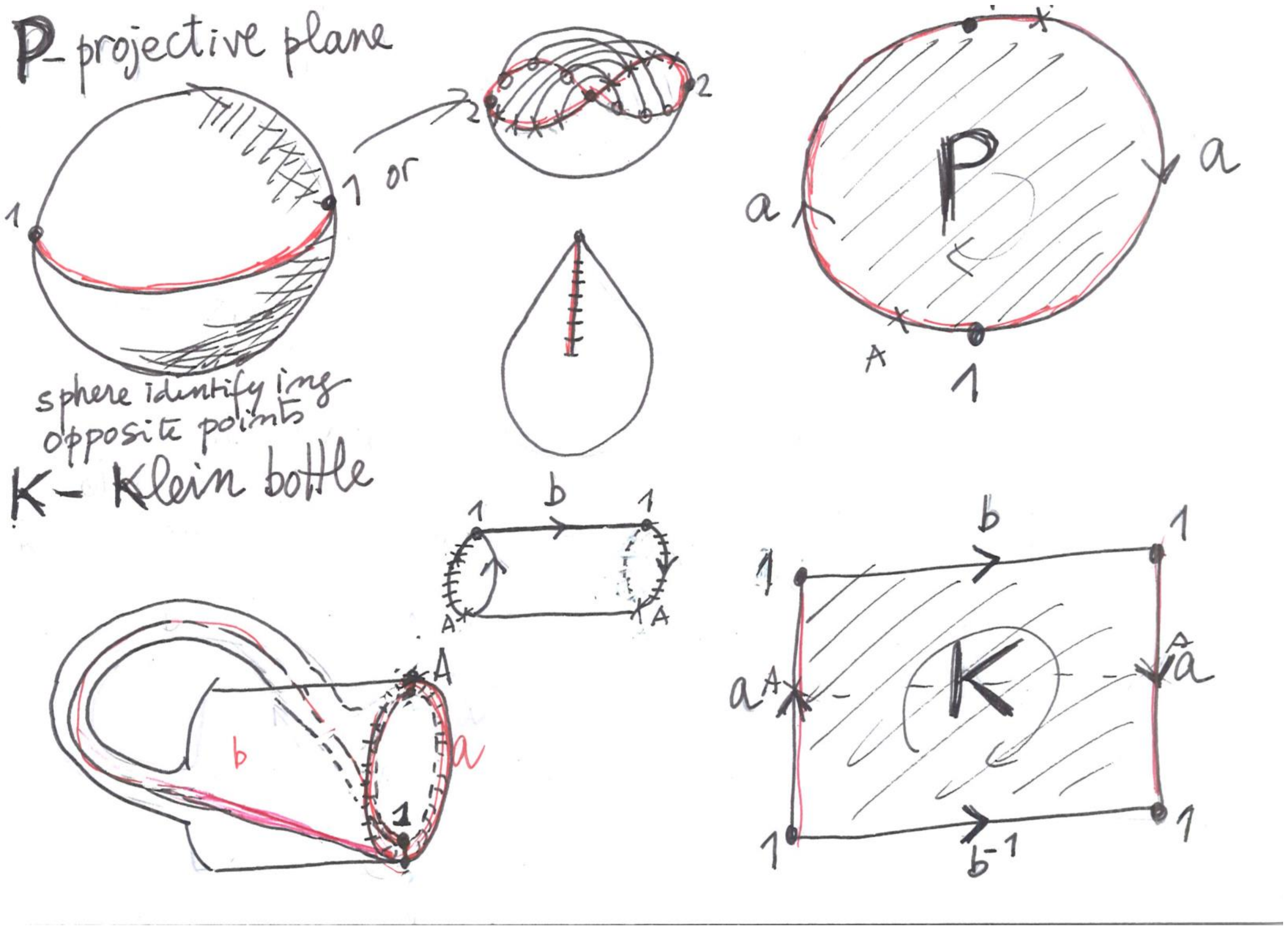
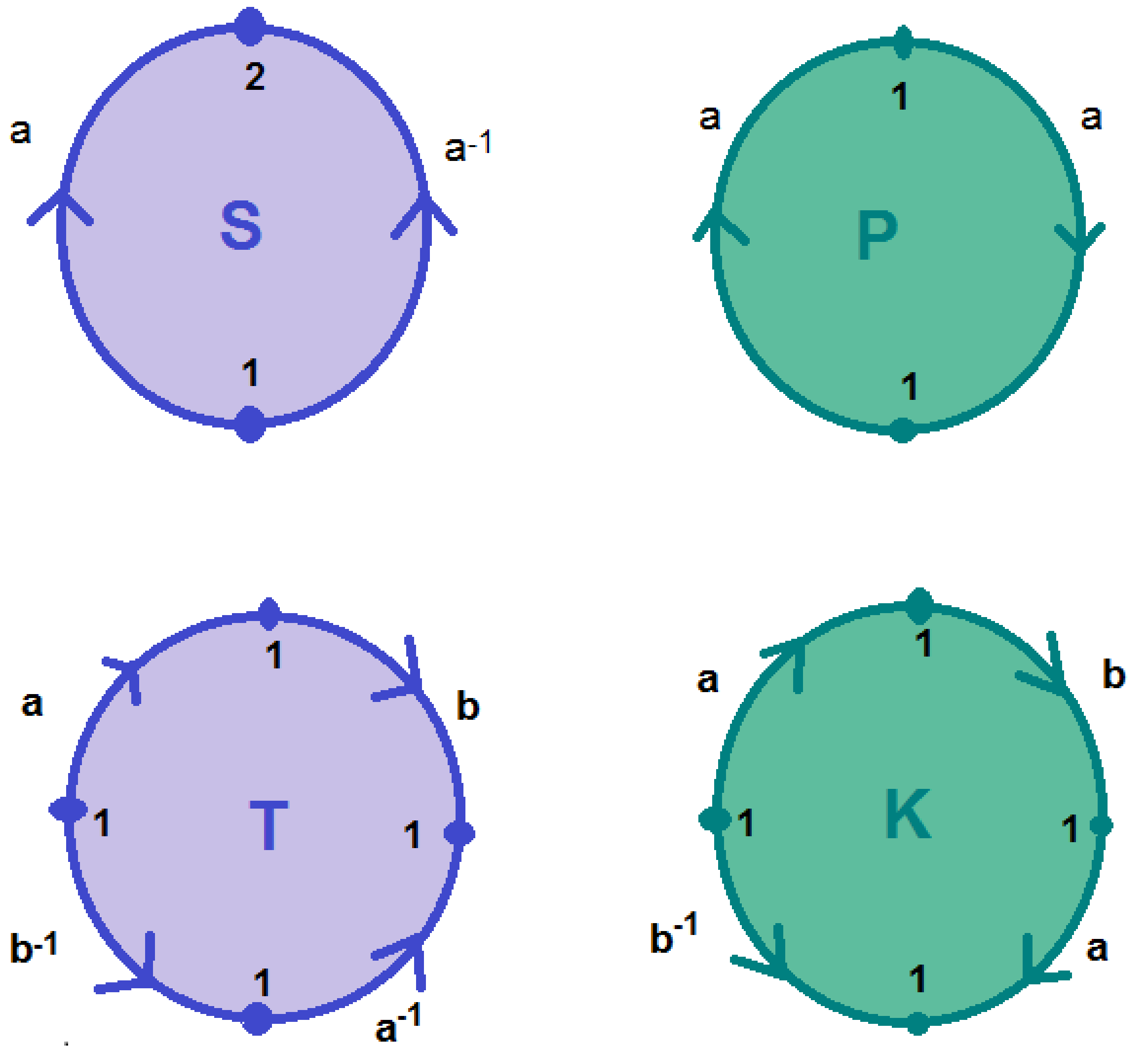
NON-ORIENTABILITY OF THE SURFACE

Construction of a non-orientable surface



Non-orientable surfaces: exemples

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2N-labelled polygons/words

A **2N-gon with the edges labelled** by N pairs of letters either x, x or x, x^{-1} forming **a word** (indicating how to "sew" the pairs of edges to recover the surface) represents **a surface**.

The extremities of the edges correspond to points on the surface that we number 1,2, … and are called the **vertices of the plane model** (not of the polygon!)

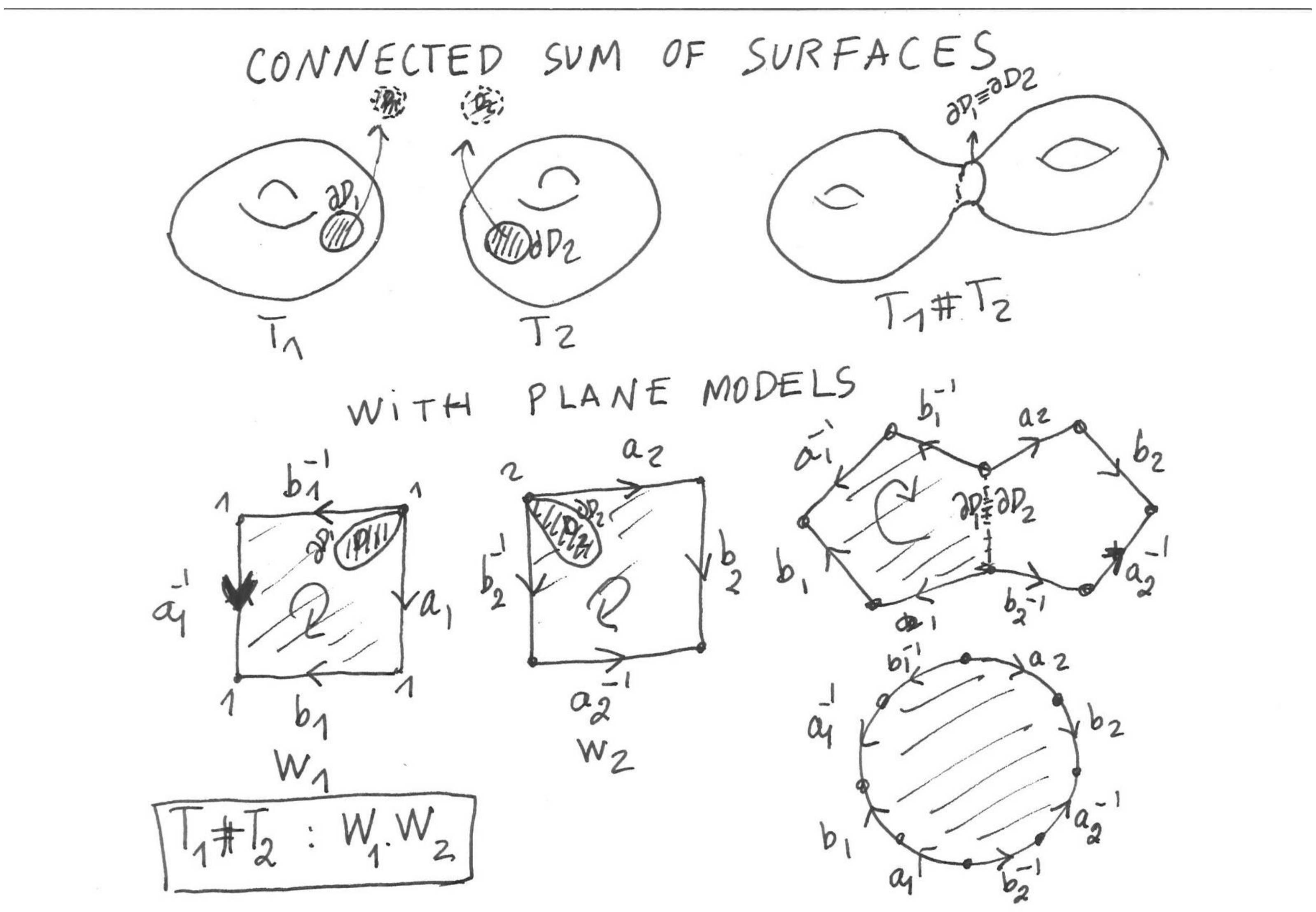
Exercise : 1) Represent the plane models described by the words:

S_1 : $\mathbf{abcdc^{-1}b\,a^{-1}\,e\,d^{-1}\,e^{-1}}$ **and** S_2 : $\mathbf{abc\,d^{-1}\,e^{-1}c^{-1}b^{-1}a^{-1}\,de}$

2) Do they represent an orientable or a non-orientable surface? Why?
3) Determine the number of edges and vertices of the plane model.

We prove next, by defining elementary operations on words that do not change the surface represented , that any word with N pairs of letters x, x or x, x^{-1} has a **normal form** corresponding to a surface. This proves

Connected sum of surfaces



Elementary transformations on a word W that do not change the (class of homeomorphism of the) represented surface

1) cyclic permutation of the letters

$$a_1 b_1 a_2 b_2 a_2^{-1} b_2^{-1} \; a_1^{-1} \; b_1^{-1} = a_1^{-1} \; b_1^{-1} a_1 b_1 a_2 b_2 a_2^{-1} b_2^{-1}$$

2) Introduction or removal of a sequence $\mathbf{x\,x^{-1}}$ in words with more then one pair of letters.

3) $\mathbf{A\,x\,B\,C\,x^{-1}} \Rightarrow \mathbf{A\,x\,C\,B\,x^{-1}}$ **orientable case**

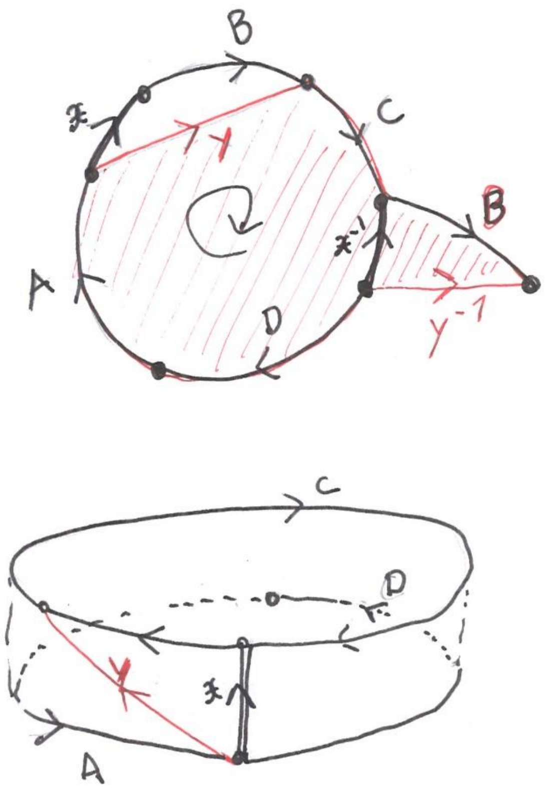
4) $\mathbf{A\,x\,B\,x\,C} \Rightarrow \mathbf{A\,y\,y\,B^{-1}\,C}$ **non orientable case**

if $B = b_1 \ldots b_k$ then $\mathbf{B^{-1}} = b_k^{-1} \ldots b_1^{-1}$.

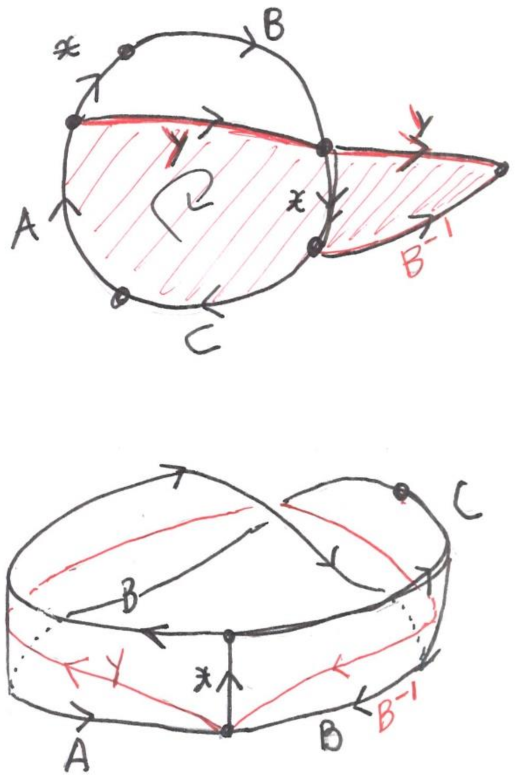
5) Replace the word **W** by $\mathbf{W^{-1}}$

Elementary operations 3 and 4

$A\,x\,B\,C\,x^{-1} \rightarrow A\,x\,C\,B\,x^{-1}$



$A\,x\,B\,x\,C \rightarrow A\,y\,y\,B^{-1}\,C$



From words/plane models to surfaces

1) Any **orientable word W** is either **a sphere S** or a **connected sum, mT, of m torus.**

W an **orientable word** with n-pairs of letters x, x^{-1} .

A pair **x, x^{-1}** and **y, y^{-1}** is a separated pair of W if

$$W = A\, x\, B\, y\, C\, x^{-1}\, D\, y^{-1}$$

Using operations 1 to 3 , W is transformed into

$$W \approx x\, y\, x^{-1}\, y^{-1}\, A\, D\, C\, B$$

Therefore W ≈ mT . V where V is a word with no separated pair.

If W is orientable, using oper. 2, V reduces to $V \approx xx^{-1}$.

From words/plane models to surfaces

1) Any **non-orientable word W** reduces to **W≈kP** or **W ≈ kP . mT ≈ mT. kP**

W is a **non orientable word** with n-pairs of letters x, x^{-1} or x, x with at least one pair x,x.

Using Op.s 1 and 5: W≈ xAxB ≈ xxAB

(not that easy!) Assume W ≈ tP. Q_t .

Prove that if Q_t still contains a pair x,x then W ≈ (t+1)P. Q_{t+1}

where Q_{t+1} contains two edges less then Q_t .

Conclude that W≈kP or W ≈ kP . mT ≈ mT. kP

Every word/plane model represents a surface !

Classification of compact surfaces – Theorem I

Theorem (known late XIX, Kerékjártó 1923, Rado 1924, Seifert Threfall 1934)

Let $\boldsymbol{W}$ *be a 2N- word with N-pairs of letters* a_i, a_i *or* a_i, a_i^{-1}, $i=1, \ldots, N$.

Then using the elementary operations 1) to 5) $\boldsymbol{W}$ *can be reduced to a canonical form* $\boldsymbol{W_N}$*, representing a compact surface:*

If ***W is orientable*** *(no pair* a_i, a_i*) then*

Either $\boldsymbol{W_N} = a_1 a_1^{-1}$ *representing a Sphere* ***S***

Or $\boldsymbol{W_N} = \underline{a_1 a_2 a_1^{-1} a_2^{-1}}\ \underline{a_3 a_4 a_3^{-1} a_4^{-1}} \ldots \underline{a_{m-1} a_m a_{m-1}^{-1} a_m^{-1}} = \boldsymbol{m\,T}$

If ***W is non-orientable*** *(a pair* a_i, a_i*) then*

Either $W_N = a_1 a_1$ *representing a (Real) Projective Plane* ***P***

Or $\boldsymbol{W_N} = \underline{a_1 a_2 a_1^{-1} a_2^{-1}} \ldots \underline{a_{s-1} a_s a_{s-1}^{-1} a_s^{-1}}\ \underline{b_1 b_1} \ldots \underline{b_k b_k} = \boldsymbol{s\,T.\,kP = mT.\,P}$ *or* $\boldsymbol{mT.K}$

Exercise 2:

1) Prove that the connected sum of two projective planes is a Klein bottle: P.P ≈K.

2) Prove that the connected sum of two projective planes is a Klein bottle: PPP ≈ PK ≈ TP

3) Conclude equality of the Classification of compact surfaces

Exercise 3 : Reduce the words of Exercise 1 to the normal form with the elementary operations and classify the surfaces :

S_1 : $\mathbf{abcdc^{-1}\,b\,a^{-1}\,e\,d^{-1}\,e^{-1}}$

S_2 : $\mathbf{abc\,d^{-1}\,e^{-1}\,c^{-1}b^{-1}a^{-1}\,de}$

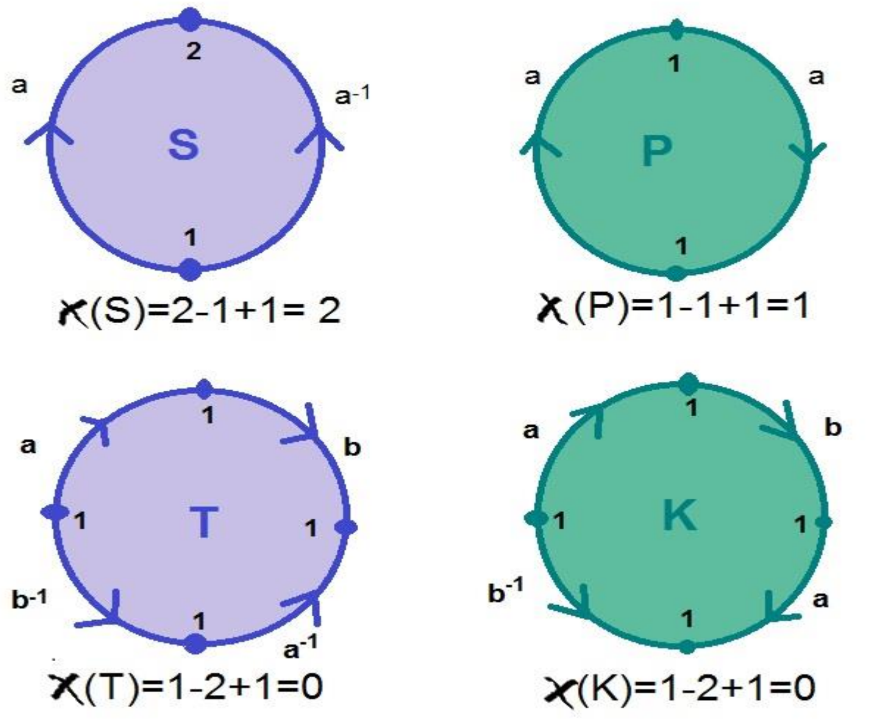
Euler – Poincaré characteristic of a surface

- Euler – Poincaré Characteristic of a plane model - W
- Given a plane model W of a surface, a labelled 2N-gon, let
- N:= nº of pairs of edges
- v:= nº of vertices of the labelling
- Then the Euler- Poincaré characteristic of the plane model is:
- $\boxed{\chi(W) = v - N + 1}$

Proposition

1) ***The Euler-Poincaré characteristic*** *of a plane model W remains invariant under the elementary operations and is, therefore,* ***an invariant of the surface.***
2) ***The Euler-Poincaré characteristic*** *of a* ***connected sum*** *of two surfaces is given by:*

$$\chi(W1 + W2) = \chi(W1) + \chi(W2) - 2$$



$$\chi(S) = 2$$

$$\chi(mT) = 2 - 2m$$

$$\chi(P) = 1$$

$$\chi(P + mT) = 1 - 2m$$

$$\chi(K + mT) = -2m$$

Classification of compact surfaces – Theorem II

Theorem

Two surfaces S1, S2 are homeomorphic if and only if

1) $\chi(S1) = \chi(S2)$

and

2) Either both S1 and S2 are orientable

or both S1 and S2 are non-orientable

Recall

Exercise 3 : Reduce the words of Exercise 1 to the normal form with the elementary operations and classify the surfaces :

S_1 : $\mathbf{abcdc^{-1}\,b\,a^{-1}\,e\,d^{-1}\,e^{-1}}$

S_2 : $\mathbf{abc\,d^{-1}\,e^{-1}\,c^{-1}b^{-1}a^{-1}\,de}$

Now,

Exercise 4: Classify the surfaces of Exercise 1, via Classification Theorem II, via orientability + Euler-Poincaré characteristic of the surface.

Looking back at our proof

(1) Labbled 2n-polygon /word model of a surface

(2) Via Elementary operations that keep the homeomorphism class of the plane model, we proved that any labelled polygon represents a surface that is either orientable and in this case S or nT , Or non-orientable and in this case P+nT or K+nT

(3) We defined the Euler-Poincaré characteristic of a plane model, observed its invariance under the elementary operations and its behaviour under connected sum. Concluded that Orientability + Euler-Poincaré characteristic a fast way of identifying a plane model.

We may accept the theorem is proved for surfaces represented by plane models.

What remains to be verified:

(1) Does every compact surface have a plane model ? YES ! But not at all obvious.

Proof : every compact surface has **a triangulation, i.e.** is homeomorphic to a surface built up by juxtaposition of triangles along their edges.

(2) Are all the types of surfaces described non-homeomorphic? YES ! But not at all obvious.

Proof: homeormorphisms preserve orientability and the Euler-Poincaré characteristic (the normal word representing a plane model encodes the **fundamental group of the surface**).

Note: Poincaré conjecture concerns the fundamental group of the sphere.

End for today