Euler- Poincaré Characteristic,
in the frontier between topology, geometry and combinatorics

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Summary of the course

1 – Euler-Poincaré characteristic and the classification of compact - surfaces

2/3 – Applications and further consequences of the classification theorem and Euler - Poincaré characteristic

- Complexes and the Euler-Poincaré characteristic

  The deficiency theorem for Complexes with convex polygons

- regular complexes ;
- colourings
- Graph embeddings

Complexes on a surface

Polygon is a set of points homeomorophic to a disc with a finite number of points marked on the boundary. The points on the boundary are the **vertices** and the arcs between the points the **edges** of the polygon.

Complex on a surface – a mosaic of a (finite) number of polygons covering the surface (or decomposition of the surface into polygons), such that:

(C1) Each polygon has at least 3 **edges** .

(C2) Polygons either do not meet or meet along a commun edge or in a commun **vertex**

(C3) Each edge is the intersection of exactly 2 polygons

(C4) Each vertex is the meet of at least 3 edges.

The **faces** of the complex are the polygons.

Examples: **triangulations**, convex polyhedra as complexes on the sphere, examples on plane models.

Complexes on a surface

Theorem (Euler-Poincaré formula for complexes)

Let **C** be a finite complex on a surface $\mathcal{S}$. Consider:

f = nº of faces (polygons) of C
e = nº of edges of C
v =nº of vertices of C

Then **v – e + f = χ($\mathcal{S}$) = χ(C)** the E-P characteristic of **C.**

Idea of the proof.

Proposition :

Every complex **C** has a **dual complex C***.
v* = f , f*= v and e*= e

Examples

Euler-Poincaré characteristic, a Theorem of Descartes, and Gauss-Bonnet Theorem

Polihedron –a complex whose faces are convex polygons and has E-P characteristic 2.

Deficiency of a polyhedron P at a vertex V: def(V)= $2\pi - \sum_{\text{F face that contains V}} \text{angle}(F_V)$

Descartes Theorem :

$$\boxed{\sum_{\text{V vertex of } \mathbf{P}} \text{def}(V) = 4\pi}$$

Generalizes to every complex C whose faces are convex polygons : **(proved)**

$$\boxed{\sum_{\text{V vertex of } \mathbf{P}} \text{def}(V) = 2\pi\,\chi(\mathbf{C})}$$

A discrete version of Gauss-Bonnet theorem: $\int_S \mathbf{K}\ \mathbf{dA} = 2\pi\,\chi(S)$, K- Gaussian curvature

Regular Complexes on a surface

A **regular complex** is a complex **C** whose vertices all have the same degree **d** and whose faces all have the same number of edges **d*.**

From Euler-Poincaré formula regular complexes on the basic surfaces S, P, T and K must verify: **(deduced** from : d**v**=2**e**, d***f**=2**e** and **v-e+f** = **v**(1-d/2 +d/d*)=**χ(C)**)

On S : (d,d*) = (3,3), (3,4), (3,5) , (4,3), (5,3) .

On P : (d,d*) = (3,3), (3,4), (3,5) , (4,3), (5,3) .

On T : (d,d*) = (3,6), (4,4), (6,3) .

On K : (d,d*) = (3,6), (4,4), (6,3)

Are all these pairs realizable as complexes?

Regular Complexes on a surface

On S : all are realizable:

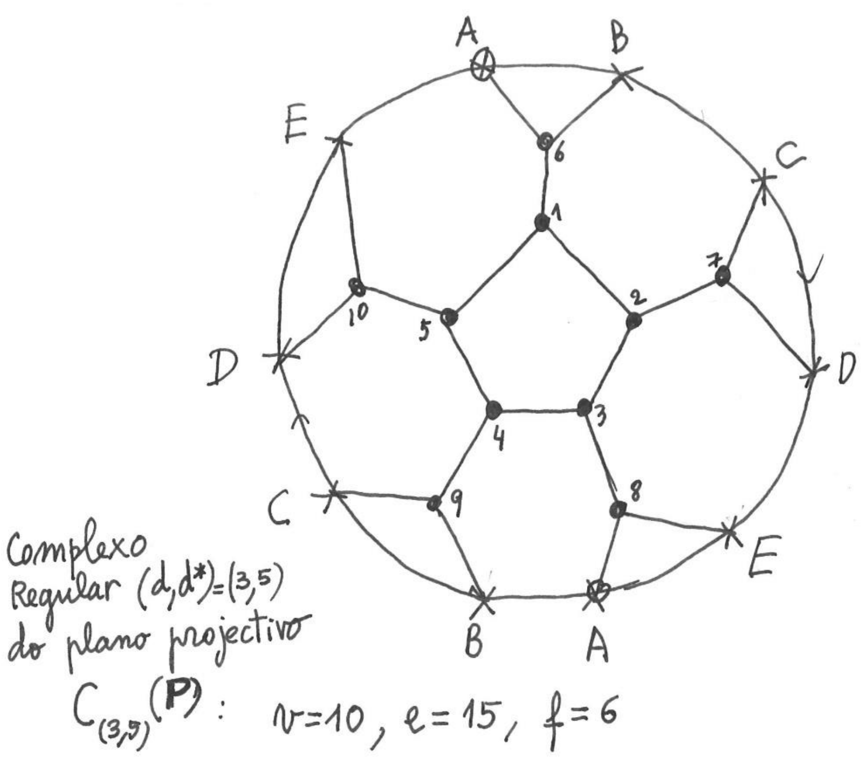
(d,d*)= (3,3) tetrahedron**, (3,4)** cube **, (3,5)** dodecahedron**,**

(4,3) octahedron**, (5,3) icosahedron.**

On P : (d,d*)= ~~(3,3), (3,4),~~ (3,5), ~~(4,3)~~, (5,3).

On T : (d,d*) = (3,6), (4,4), (6,3) . Infinite families of complexes in each case

On K ~~: (3,6), (4,4), (6,3).~~ Coxeter, Moser 1965



Exercise 5: a) In order to obtain a **complexe of P** with (d,d*)= (5,3) draw the dual of the complex **$C_{(3,5)}$ (P),** represented above.

b) The existence of infinite families of regular **complexes on T** with (d,d*)= (4,4), (3,6) or (6,3) .

Day 3

Recalling: Complexes on a surface

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Colourings of complexes (maps) on surfaces

A **colouring of a complex** on a surface is an assignement of a colour to each face so that adjacent faces have different colours.
A **complex is N-colourable** if it can be coloured with N colours.

Given a surface $\mathcal{S}$ we denote by N($\mathcal{S}$) **the minimum number of colours needed to colour ANY(every) complex on $\mathcal{S}$.**

The 4-colour conjecture (Guthrie, 1870): 4 colours are enough to colour any complex on the sphere.
Proved computationally 1970's Appel and Hanken, 1990's Robinson and Seymour

The Heawood conjecture (Heawood, 1890): For every compact surface $\mathcal{S}$

$$N(\mathcal{S}) := [(7 + \sqrt{49 - 24\,\chi(\mathcal{S})})/2\ ]$$

Proved true except for the Klein bottle (whose colouring number is 6 , not 7) in the late 1960's by Ringel and Young

χ	2	1	0	-1	-2	-3	-4	-5
N($\mathcal{S}$)	4	6	7	7	8	9	9	10

How Heawood arrived to his conjecture

Observation 1) The average of the number of edges per face of a complex C(S) , 2e/f, must be related to the number of colours needed to colour C(S).

Observation 2) There is un upper bound for the average number 2e/f depending exclusively on the surface:

$$2e/f \leq 6 (1 - \chi(S)/f)$$

Observation 3) If N is a natural number such that 2e/f <N for every map on the surface S then, every map on the surface S is N-colorable and therefore $N(\mathcal{S}) \leq N$.

Theorem (Heawood 1890)

1) Every complex on the Sphere or Projective plane is 6-colorable.

2) Every complex on the Torus or Klein bottle is 7-colorable.

3) Every complex on a surface S of negative Euler characteristic can be coloured with

$$N(\mathcal{S}) \leq [(7 + \sqrt{49- 24\, \chi(\mathcal{S})})/2\ ] \text{ colours.}$$

some further topics on Combinatorics and Topology of surfaces

Graphs embeddings: Given a graph G=(V,E) find a minimal surface for G is the orientable surface S, with maximum $\chi(S)=\Upsilon(G)$, where G can be "drawn" so that the edges only meet at commun vertices.

Theorem (Kuratowski 1930's): a graph G is **planar** i.e. can be embedded in the sphere or in the plane iff contains no subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$.

Theorem (Steinitz 1930's): a graph G is the skeleton of a 3-dimensional convex polyhedron iff it is 3-connected.

Theorem (Ringel, Young 1960's) (relevant for Ringel, Young's proof of Heawood conjecture by)

1) $\Upsilon(K_n) = 2\,[n(7-n)/12]$
2) $\Upsilon(K_{m,n}) = 2\,[(m+n+ mn/2)/2]$

Theorem (Robinson-Seymour 1990's): There is a finite number of minimal graphs to exclude for embeddability in any surface.

Exercise 5: Prove that in order to colour the regular complex $\mathbf{C_{(3,5)}}$ **(P) of the projective** plane one needs 6 colours.
This remark together with Heawood Theorem implies that **N(P)**=6.

Exercise 6: Draw the complete graphs K_5 embedded in the torus.
Note that by Ringel Theorem K_6 and K_7 can also be embedded in the torus. Try !

Two nice problems on planar graphs:

Sylvester problem (for more on this game see [IS] *): Consider a finite set of points in the plane, not all in the same line. Prove that there is one line that contains exactly 2 of the given points.*

Game of sprouts (Paterson, Conway, see [WW]): ***For 2 players playing alternately:***

Draw N points(dots) in the plane.

Each player draws at his turn a curve connecting two spots, or a spot to itself without crossing curves already drawn. Once the curve is drawn a new spot is placed on that curve.

No spot may have more then three lines ending at it.

The player that cannot play looses.

Prove that the game ends at most in 3N-1 moves and lasts at least 2n moves (Conway, Mollison).

Winning strategies? (the game may be played on any surface not just in the plane!)

Overview of the course

1) **Compact surfaces as plane models** – labelled polygons/words

Elementary transformations/ connected sum of surfaces

Classification theorem 1 - reduction to a normal word W_c

definition of the E-P characteristic of a word $\chi(W)$

Classification theorem 2 - W + $\chi(W)$

2) **Compact surfaces as complexes C**

the E-P charactheristic **$\chi(C)$: = v – e+ f =** the E-P characteristic of any plane model of **C.**

dual complex C*

Generalization of the deficiency theorem of Descartes for convex polihedra to complexes whose faces are convex plane convex polygons, a discrete version of Gauss- Bonnet theorem**.**

3) **Problems that can be attacked with topological methods**

(finite) **Regular complexes** of surfaces with **$\chi(C) \geq 0$**

Colourings of complexes - 4 colour and Heawood conjectures.

A relevant topic in topology/combinatorics: Embedding graphs on surfaces

Two nice problems on planar graphs

4) Surfaces and the fundamental group – Algebraic topology

Poincaré: in order to get information about the topology of a surface e.g. (d-dimensional manifold), study the loops (closed walks) starting and ending at a point (any point) in the surface.

The fundamental group $\pi(S)$ of a d-dimensional manifold is a group generated by loops that are not equivalent, i.e. that can not **be continuosly transformed one into the other within the surface** (k- manifold).

For a **compact surface** S **we get a presentation of its fundamental group** , $\pi(S)$, **directly from any plane model/ word W.** Different word models describe **isomorphic** groups.

Poincaré conjecture : The sphere S^3 of dimension 3 is the unique compact 3-dimensional manifold with trivial fundamental group.

(W. Thurston 70's, R. Hamilton 80's , proved by G. Perelman 2006)

See the page of the Millenium Problem of the Clay Institute:
http://www.claymath.org/millennium-problems/poincaré-conjecture

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