An Investigation with two bodies but no crime:
The Kepler Problem
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A cautionary note

These slides are an incomplete account of the investigations. More details were given only on the blackboard. However, using the references you should be able to reconstruct the missing parts.

A tentative Plan

- **Lecture 1**: One body disappears
  Going in circles
- **Lecture 2**: Prison Break
  One body is found
- **Lecture 3**: It all ends in the Spring

Bibliography

- *On the geometry of Kepler Problem*, J. Milnor, The American Mathematical Monthly
- *Feynman's Lost Lecture. The motion of the Planets around the Sun*, D.L. Goodstein and J.R. Goodstein
- *Introducción a la Mecánica Celeste*, R. Ortega and A. Ureña
- *Classical Mechanics with Calculus of Variation and Optimal Control. An intuitive introduction*, Mark Levi
- *Huygens and Barrow, Newton and Hooke*, V. I. Arnold

LECTURE I.
The Kepler Problem and the Two Body Problem

From Newton's Second law of Dynamics

$$F = ma \quad (a = \ddot{x})$$

plus Newton's universal law of gravitation

$$F = -\lambda \frac{x}{|x|}, \qquad \lambda = |F| = GMm\frac{1}{|x|^2}$$

we obtain:

The Kepler Problem (KP)

$$\ddot{x} = -GM\frac{x}{|x|^3}, \quad x \in \mathbf{R^3} \setminus \{\mathbf{0}\}$$

almost.....this is an approximate model!

2BP

Two body problem (2BP):

$$(x, m_1),\ (y, m_2),\ x \neq y \in \mathbf{R^3}$$

$$\ddot{x} = -Gm_2\frac{x-y}{|x-y|^3}, \quad \ddot{y} = -Gm_1\frac{y-x}{|y-x|^3}$$

Center of mass of the system:

$$C(t) = \frac{m_1}{m_1+m_2}x(t) + \frac{m_2}{m_1+m_2}y(t)$$

$$\ddot{C} = 0 \Longrightarrow C(t) = \alpha t + \beta, \quad \alpha, \beta \in \mathbf{R^3}.$$

2BP: looking at the system from its center

Setting

$$\tilde{x} = x - C(t), \ \tilde{y}(t) = y - C(t),$$

we have $\tilde{C}(t) = \frac{m_1}{m_1+m_2}\tilde{x}(t) + \frac{m_2}{m_1+m_2}\tilde{y}(t) = 0$, so that

$$m_1\tilde{x} + m_2\tilde{y} = 0$$

Then

$$\tilde{x} = -\frac{m_2}{m_1}\tilde{y}, \ \ \tilde{y} = -\frac{m_1}{m_2}\tilde{x}$$

and substituting in

$$\ddot{x} = -Gm_2\frac{x-y}{|x-y|^3}, \quad \ddot{y} = -Gm_1\frac{y-x}{|y-x|^3},$$

we have a decoupled system of two KPs!

$$\ddot{\tilde{x}} = -\frac{Gm_2^3}{(m_1+m_2)^2}\frac{\tilde{x}}{|\tilde{x}|^3}, \quad \ddot{\tilde{y}} = -\frac{Gm_1^3}{(m_1+m_2)^2}\frac{\tilde{y}}{|\tilde{y}|^3}$$

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New solutions from old ones: symmetries

If $x(t)$ is a solution of (KP) then

- $$t \to x(-t)$$
  is also a solution
- if $R \in M_3$ is an orthogonal matrix ($R^tR = I$) then
  $$Rx(t)$$
  is also a solution. In particular rotating a solution ($R \in SO(3), \det R = +1$) we still get a solution

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Some conserved quantities of (KP)

The energy:

$$E = \frac{|\dot{x}|^2}{2} - \frac{1}{|x|}$$

The angular momentum

$$h = x \wedge \dot{x}$$

The conservation of h implies that given a solution for which $h(0) \neq 0$ we can choose a coordinate system where h points in the positive x_3 direction (then $h = (0, 0, |h|)$) and introduce polar coordinates in the plane of the solution. Then

$$x(t) = r(t)(\cos\theta(t), \sin\theta(t), 0)$$

$$|h| = r^2(t)\dot{\theta}(t)$$

Next result deals with solutions such that $h \neq 0$.

Velocity circles and orbits: a theorem by Hamilton (from Milnor's article)

Theorem (Hamilton, 1846)

As t varies, the velocity vector $v = \frac{dx}{dt} = \dot{x}$ moves along a circle C which lies in some plane containing the origin O.

Any such circle can occur and this "velocity circle" together with its orientation determines the orbit $x = x(t)$ uniquely

The corresponding orbit is:

- an ellipse if O is inside the velocity circle
- a parabola if O is on the velocity circle
- an hyperbola if O is outside the velocity circle

We're losing time...

From the conservation of the angular momentum we deduce that the function $t \to \theta(t)$ is a diffeomorphism (between open intervals) and that its inverse $t = t(\theta)$ satisfies

$$\frac{dt}{d\theta} = \frac{r^2(t(\theta))}{|h|}.$$

We denote by:

$$r(\theta) = r(t(\theta)), \quad x(\theta) = x(t(\theta)) = r(\theta)(\cos\theta, \sin\theta, 0), \;\; v(\theta) = v(t(\theta))$$

The velocity circle

Then, from

$$\frac{dv}{dt} = -\frac{x}{|x|^3}$$

we get

$$\frac{dv}{d\theta} = R(-\cos\theta, -\sin\theta, 0), \;\; R = \frac{1}{|h|}$$

Integrating we get

$$v(\theta) = R(-\sin\theta, \cos\theta, 0) + \mathbf{c}$$

Note that

- $v(\theta)$ belongs to a circle with center **c** and radius R
- $x(\theta) \perp v(\theta) - \mathbf{c}$

Velocity circle (with $\epsilon < 1$)

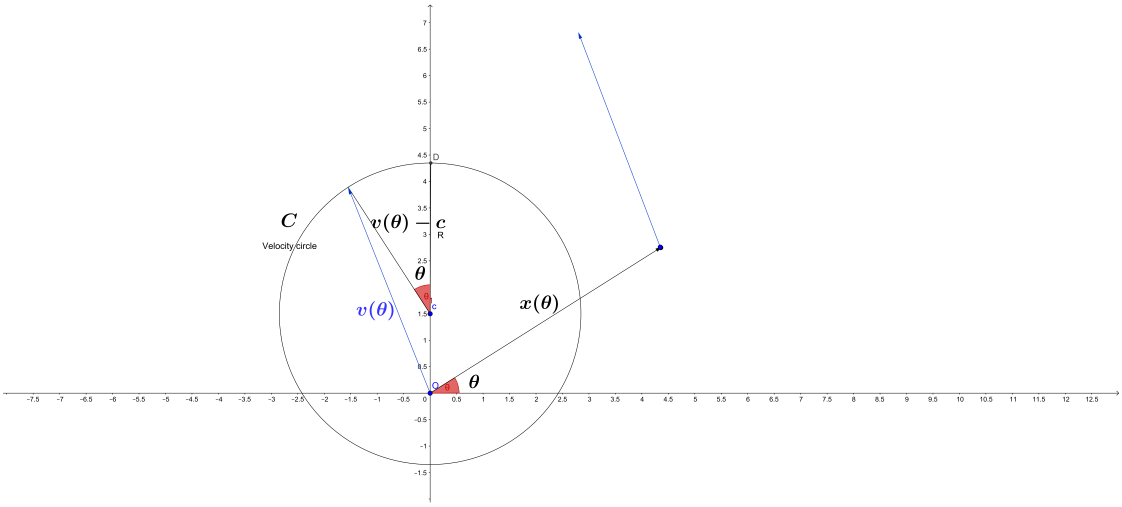

Figure 1: Without loss of generality, we can assume **c**$=(0, c, 0)$ with $c > 0$. Then, C: $v(\theta) = R(-\sin\theta, \cos\theta + \epsilon, 0)$ where $R = \frac{1}{|h|}$ and $\epsilon = \frac{|c|}{R}$. In the figure $\epsilon < 1$

Feynman's proof that the orbits are ellipses

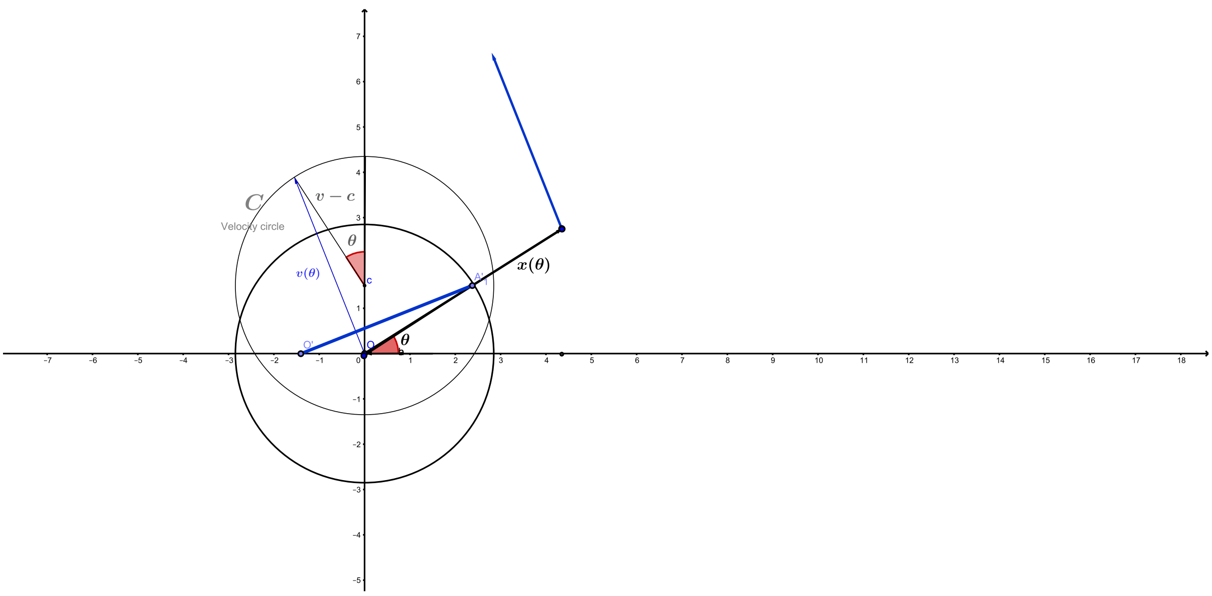

Figure 2: the velocity circle is rotated of $-\frac{\pi}{2}$ and the vertex of the angle is translated to the origin

Feynman's proof that the orbits are ellipses

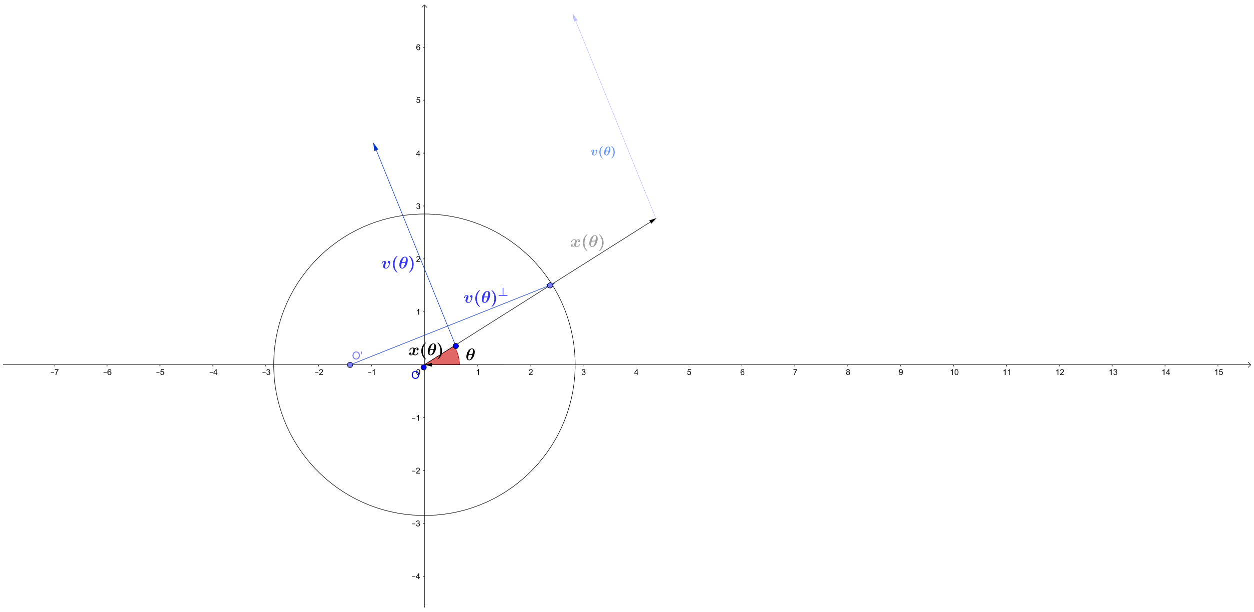

Figure 3: The trajectory is then suitably rescaled: $x(\theta)$ belongs now to the perpendicular bisector of $v(\theta)^\perp$. The Geogebra file "Feynman's construction of the ellipse" shows how it works

Back to Milnor's paper: solutions of (KP) are contained in conic sections

$$|h| = |x(\theta) \wedge v(\theta)| = |r(\cos\theta, \sin\theta, 0) \wedge R(-\sin\theta, \epsilon + \cos\theta, 0)| =$$
$$= rR(1 + \epsilon\cos\theta)$$

$$r = r(\theta) = \frac{|h|^2}{1 + \epsilon\cos\theta}, \quad 1 + \cos\theta > 0$$

polar equation of a conic section with eccentricity ϵ and a focus at O. $\theta = 0$ corresponds to the closest approach to the focus (pericenter)

À la recherche du temps perdu

Given c and R, we know

$$\epsilon = \frac{|c|}{R} \text{ and } |h| = \frac{1}{R}$$

so that

$$r(\theta) = \frac{|h|^2}{1 + \epsilon \cos\theta}$$

is uniquely determined. To show that any conic gives a solution of (KP) we go back, retracing our steps and defining the t variable as

$$t = t(\theta) = \int_0^\theta \frac{r^2(\phi)}{|h|}\, d\phi.$$

Considering the inverse function $\theta = \theta(t)$ a computation shows that

$$x(t) = r(\theta(t))(\cos\theta(t), \sin\theta(t), 0)$$

satisfies (KP)

Energy along an orbit

Computing the energy

$$E = \frac{|v|}{2} - \frac{1}{|x|}$$

at $\theta = 0$ and recalling that $\epsilon = \frac{c}{R}$ and $R = \frac{1}{|h|}$ we get

$$|v(0)| = (c + R)^2$$

$$|x(0)| = \frac{|h|^2}{1+\epsilon} = \frac{1}{R(1 + cR)}$$

so that

$$2E = c^2 - R^2 = \frac{1}{|h|^2}(\epsilon^2 - 1)$$

We conclude

- $E < 0 \iff$ the orbit is elliptic
- $E = 0 \iff$ the orbit is parabolic
- $E > 0 \iff$ the orbit is hyperbolic

Elements of an ellipse and energy. Kepler's third law

Elements of the ellipse:

$$2a = r_{min} + r_{max} = \frac{|h|^2}{1-\epsilon^2}$$

$$b = y_{max} = \frac{|h|^2}{\sqrt{1-\epsilon^2}} = a\sqrt{1-\epsilon^2}$$

$$2E = \frac{1}{|h|^2}(\epsilon^2 - 1) = -\frac{1}{a}$$

From Kepler's second law, the period T satisfies

$$\text{Area of the elipse} = \pi ab = \frac{|h|}{2}T$$

so that we have Kepler's third law:

$$T = 2\pi a^{\frac{3}{2}}$$

LECTURE II.
(KP): The associated vector field

(KP) is equivalent to the first order system

$$(KP_s) \qquad \dot{x} = v, \quad \dot{v} = -\frac{x}{|x|^3}$$

with $(x, v) \in \Omega := (\mathbf{R^3} \setminus \{\mathbf{0}\}) \times \mathbf{R^3}$
The vector field

$$F(x, v) = \left(v, -\frac{x}{|x|^3}\right)$$

is smooth on $\Omega \implies$ for any $t_0 \in \mathbf{R}$, $(\mathbf{x_0}, \mathbf{v_0}) \in \mathbf{\Omega}$ there exists only one solution of $(KP)_s$ such that $x(t_0) = x_0,\ v(t_0) = v_0,$ defined on a (maximal) interval $J =]\alpha, \omega[$. Solutions are invariant for translations in time. If $(x(t), v(t))$ solve the previous Cauchy problem, on J, then $(x(t-t_0), v(t-t_0))$ solve the system with $x(0) = x_0,\ v(0) = v_0$ on $J - t_0$

Going global

Theorem Solutions with $h(0) = x(0) \wedge v(0) \neq 0$ are defined on $\mathbf{R}$.

Consequence of the following result from the general theory of ODEs

Theorem (Escape from Compact Sets) Consider the differential equation

$$X' = F(X),$$

where $F : \Omega \subset \mathbf{R^n} \to \mathbf{R^n}$ is continuous and Ω is open. If a maximal solution $X = X(t),\ t \in]\alpha, \omega[$ verifies $\omega < +\infty$, then one of the following alternatives holds:

- $$|X(t)| \to +\infty \ \ as \ \ t \to +\omega$$
- there exists a sequence of points $t_n \to \omega$ such that

$$\text{dist}(X(t_n), \partial\Omega) \to 0$$

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Going global, periodically

The fact that a solution with nonzero angular momentum and negative energy (elliptic orbit) is actually time periodic is guaranteed by the following general result:
Theorem Consider the differential equation

$$X' = F(X),$$

where $F : \Omega \subset \mathbf{R^n} \to \mathbf{R^n}$ is locally Lipschitz continuous and Ω is open. If a solution $X : [0, p] \to \mathbf{R^n}$ satisfies

$$X(0) = X(p)$$

then its maximal domain is $\mathbf{R}$ and the corresponding extension is p periodic.

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Kepler's equation: the importance of being eccentric

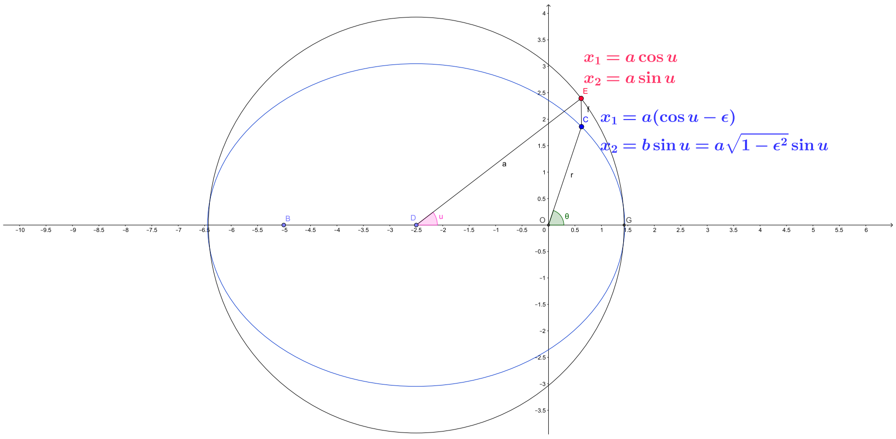

Figure 4: A suitable parametrization of the ellipse

Kepler's equation: the importance of being eccentric

Using the eccentric anomaly u to parametrize an elliptic orbit we get:

$$x = (x_1, x_2) = a(\cos u - \epsilon, \sqrt{1-\epsilon^2}\sin u)$$

Since

$$|h| = x_1\dot{x}_2 - x_2\dot{x}_1$$

along a solution $u(t)$ satisfies

$$a^2\sqrt{1-\epsilon^2}(\dot{u} - \dot{u}\epsilon\cos u) = |h|$$

Integrating from t_0 such that $u(t_0) = 0$

$$u - \epsilon\sin u = \frac{|h|}{a^2\sqrt{1-\epsilon^2}}(t - t_0)$$

which leads to Kepler's equation

$$u - \epsilon\sin u = \frac{2\pi}{T}(t - t_0)$$

Solving Kepler's equation with the help of Newton

Fixed any time t, we want to solve

$$f_\epsilon(u) = u - \epsilon \sin u = \frac{2\pi}{T}(t - t_0)$$

Let $\xi = \frac{2\pi}{T}(t - t_0)$ be fixed. It is sufficient to consider $\xi \in]0, \pi[$

The equation

$$f_\epsilon(u) = u - \epsilon \sin u = \xi$$

can be efficiently solved using Newton's method:

$$u_{n+1} = u_n - \frac{f_\epsilon(u_n) - \xi}{f'_\epsilon(_n)}$$

starting from a suitable u_0. What does it mean 'suitable'? One can prove that:

Newton's method converges to for any u_0 such that $u_1 \in [0, \pi]$

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Solving the Kepler equation with the help of Bessel

$$f_\epsilon(u) = u - \epsilon \sin u = \frac{2\pi}{T}(t - t_0)$$

Let $\xi = \frac{2\pi}{T}(t - t_0)$.
Since f_ϵ is strictly increasing and such that $f_\epsilon(u) - u$ is 2π periodic and odd, there exist an inverse function $u = K\epsilon(\xi)$ with the same properties.
Then we can develop $K_\epsilon(\xi) - \xi$ in Fourier series:

$$K_\epsilon(\xi) - \xi = \sum_{n=1}^{\infty} b_n \sin n\xi$$

with

$$b_n = \frac{2}{\pi} \int_0^\pi (K_\epsilon(\xi) - \xi) \sin n\xi \, d\xi$$

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Solving the Kepler equation with the help of Bessel

$$b_n = \frac{2}{\pi}\int_0^\pi (K_\epsilon(\xi) - \xi)\sin n\xi \, d\xi$$

Integrating by parts

$$b_n = \frac{2}{n\pi}\int_0^\pi K'_\epsilon(\xi)\cos n\xi \, d\xi$$

By the change of variables $u = K_\epsilon(\xi)$ since $u - \epsilon \sin u = \xi$ we get finally

$$b_n = \frac{2}{n\pi}\int_0^\pi \cos n(u - \epsilon \sin u)\, du = \frac{2}{n}J_n(n\epsilon)$$

where the Bessel function of order $n \geq 1$ is defined by

$$J_n(x) = \frac{1}{\pi}\cos(nu - x\sin u)$$

Solving the Kepler equation with the help of Bessel

$$K_\epsilon(\xi) = \xi + \sum_{n=1}^{\infty}\frac{2}{n}J_n(\xi)\sin n\xi, \;\; \xi \in \mathbf{R}$$

$$\xi = \frac{2\pi}{T}(t - t_0)$$

From here we get an "explicit" formula for elliptic movements (when the ellipse lies in the plane (x_1, x_2) and the major axis lies on the x_1 axis:

$$x_1(t) = a(\cos u(t) - \epsilon), \;\; x_2(t) = a\sqrt{1-\epsilon^2}\sin u(t)$$

$$u(t) = \nu(t - t_0) + \sum_{n=1}^{\infty}\frac{2}{n}J(n\epsilon)\sin n[n\nu(t - t_0)], \;\; \nu = \frac{1}{a^{3/2}}$$

To recover the position of a planet in $\mathbf{R^3}$ at a given time one has to consider the position of its orbital plane with respect to the plane containing the orbit of the Earth, and then the inclination of the major axis of the orbit of the planet in its plane. This is done introducing astronomical coordinates, which are Euler's angles.

LECTURE III.

On a collision course....no Bruce Willis will save us

Considering a suitable coordinate system, a fixed rectilinear solution ($h = 0$) satisfies the scalar equation

$$\ddot{x}_1 = -\frac{1}{x_1^2}$$

There are three types:

- ejection-collision: defined on a finite interval $]t_0, t_1[$ with

$$(ejection) \quad \lim_{t \to t_0^+} x_1(t) = 0, \quad \lim_{t \to t_0^+} \dot{x}_1(t) = +\infty$$

$$(collision) \lim_{t \to t_1^-} x_1(t) = 0, \quad \lim_{t \to t_1^-} \dot{x}_1(t) = -\infty$$

- ejection-escape: defined on a half line $]t_0, +\infty[$ with $\lim_{t \to +\infty} x_1(t) = +\infty$
- capture-collision: reverse time in the previous class

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Spring is coming....

Consider the equation

$$(H) \quad \ddot{x} = -\frac{k}{m}x, \quad x \in \mathbf{R}^2$$

of a point mass m subjected to a the force corresponding to a Hookean spring.
In what follows we set $\frac{k}{m} = 1$

(H) is actually a system of two uncoupled linear oscillators.

As for the (KP), the angular momentum of the solutions of (H) is conserved (it's a central force field), rotating a solution we still obtain a solution, and the same occurs reversing time.

Theorem 1 *Trajectories of Hooke's equation with nonzero angular momentum are ellipses centred at O. Moreover any such motion is the sum of two circular motions with angular velocities $+1$ and -1*

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This is not an epicycle!

Proof: W.l.g. we can write any solution in the form

$$z(t) = pe^{it} + qe^{-it}, \quad p > q \geq 0$$

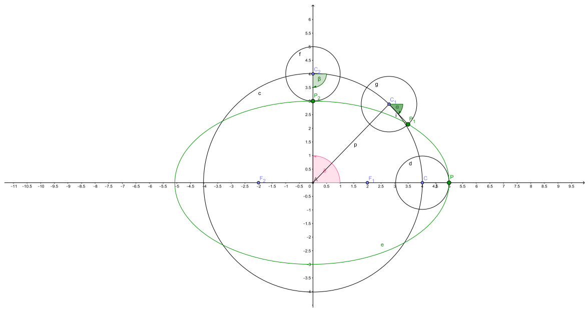

Figure 5: ellipse with center at the origin O and foci at $\pm 2\sqrt{pq}$ $a = p + q,\ b = p - q,\ \epsilon = \frac{2\sqrt{pq}}{p+q}$

Squaring an ellipse...Wait, what?

Lemma If $z(t)$ is an Hookean ellipse, then $w(t) = z^2(t)$ is an ellipse with a focus at the origin O

Proof

$$z^2(t) = \underbrace{p^2e^{2it} + q^2e^{-2it}}_{\text{ellipse centred at } O \text{ with foci at } \pm 2pq} + 2pq$$

The times they are a-changing

$w(t)$ does not satisfy the law of the areas. The angular momentum of $w(t)$ is

$$2\dot{\theta}(t)|w(t)|^2 = 2|h||z(t)|^2,$$

and is not constant. However, we have the following result:

Theorem Suppose a point in the z complex plane moves according to Hooke's law

$$\ddot{z} = -z$$

. We square z and introduce in the trajectory of the point

$$w = z^2$$

a new time τ so that the law of areas is satisfied. Then $\tilde{w}(\tau)$ satisfies the (KP):

$$\frac{d^2\tilde{w}}{d\tau^2} = -4E_H\frac{\tilde{w}}{|\tilde{w}|^3}, \quad E_H = \frac{1}{2}(|\dot{z}|^2 + |z|^2)$$

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Proof

We introduce the new time τ such that, if

$$\tilde{w}(\tau) = |z(t(\tau))|^2 e^{2i\theta(t(\tau))}$$

then

$$2\frac{d\theta}{dt}(t(\tau))t'(\tau)|z(t(\tau))|^4 = constant$$

Since $\frac{d\theta}{dt}|z|^2 = constant$, we must require

$$t'(\tau) = \frac{c}{|z(t(\tau))|^2}$$

so that, choosing $c = 1$

$$\tau'(t) = |w(t)|^2, \quad \tau = \tau(t) = \int_0^t |w(s)|^2\, ds$$

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Proof

The computations are the following, where $\tilde{z}(\tau) = z(t(\tau))$:

$$\tilde{w}'(\tau) = 2\tilde{z}\dot{z}(t(\tau))t'(\tau) = 2\frac{\dot{\tilde{z}}}{\bar{\tilde{z}}}$$

$$\tilde{w}''(\tau) = \frac{2}{z\bar{\tilde{z}}}\left(-\frac{1}{(\bar{\tilde{z}})^2}\dot{\bar{\tilde{z}}}\dot{z} + \frac{\ddot{\tilde{z}}}{\bar{\tilde{z}}}\right) = \frac{-2}{\tilde{z}(\bar{\tilde{z}})^3}(|\dot{\tilde{z}}|^2 + |\tilde{z}|^2) =$$

$$= -4E_H\frac{\tilde{z}^2}{|(|\tilde{\tilde{z}}|^2)^3} = -4E_H\frac{\tilde{w}}{|\tilde{w}|^3}$$

A final mystery

The previous construction can be inverted, and the corresponding transformation is called Levi-Civita transformation. Then, there is a bijection between the solutions of (KP) and the ones of (H). In particular, the elliptical solutions of (KP) with equal energy E_{KP} correspond to elliptical solutions of (H) with equal energy E_H and vice-versa.

What happens in the limit when we consider a continuous family of solutions of (KP) with fixed energy $E = -\frac{1}{2a}$ and whose angular momentum tends to zero?

The ellipses (of fixed axis $2a$) degenerate in a segment (of length $2a$) containing the singularity as an endpoint. This segment corresponds to an ejection-collision solution, defined on a finite interval.

We pass from solutions defined on $\mathbf{R}$ to a solution defined on a finite interval.

Something seems to be lost.....

The solution in a picture from the "regularized space". End of the investigation

In the "Hookean space" the corresponding ellipses tend to a segment which has $z = 0$ as midpoint and which carries a rectilinear solution $z(t)$ of (H) defined and smooth on $\mathbf{R}$ and which makes symmetric oscillations with respect to $z = 0$

The ejection-collision solution corresponds to just a segment of this Hookean solution $z(t)$ (from $z = 0$ to the maximum distance from 0 and back). When $z(t)$ crosses $z = 0$ and describes the second half segment of its orbit, $w(\tau) = z^2(t(\tau))$ describes a second ejection collision orbit *on the same segment* of the previous one in the (KP) space.

Then, $z(t)$ corresponds to an orbit $w(\tau)$ obtained gluing ejection-collision solutions according to a "reflection rule": after collision at time τ_0^- with velocity $-\infty$ the particle is ejected at time τ_0^+ with velocity $+\infty$ and with the same energy.

This generalized solution (which actually belong to $W^{1,2}(0, 2\pi a^{\frac{3}{2}})$ is defined on $\mathbf{R}$ and can be thought of as the "true" limit of the solutions with fixed negative energy considered above